SYZYGY: WHEN SUBMODULES ALIGN

Courtney Gibbons
Hamilton College
MathFest 2019
August 4, 2019

Syzygy

- noun. The collinear arrangement of celestial bodies.
- noun. The kernel of ∂_i in a ***free resolution*** of an R-module M.

WHAT?

Recall

Given a field F and a set V, V is an F-vector space provided:

- V is an abelian group under vector addition, and
- scalar multiplication $F \times V \to V$ has an identity scalar, is associative, and is distributive over scalar addition and vector addition.

Definition

Given a ring R and a set M, M is an R***-module*** provided:

- M is an abelian group under vector addition, and
- scalar multiplication $R \times M \to M$ has an identity scalar, is associative, and is distributive over scalar addition and vector addition.

IMAGES AND KERNELS

Let A, B, and C be three R-modules. Consider homomorphisms f and g:

$$
\begin{tikzcd}
A & B \arrow[l, "f"'] & & & C \arrow[lll, "g"'] \arrow[dl] \\
& & \ker(f) \arrow[ul] \arrow[r, phantom, "\supseteq"] & \operatorname{im}(g) &
\end{tikzcd}
$$

- $\ker(f)$ is a submodule of B
- $\operatorname{im}(g)$ is a submodule of B

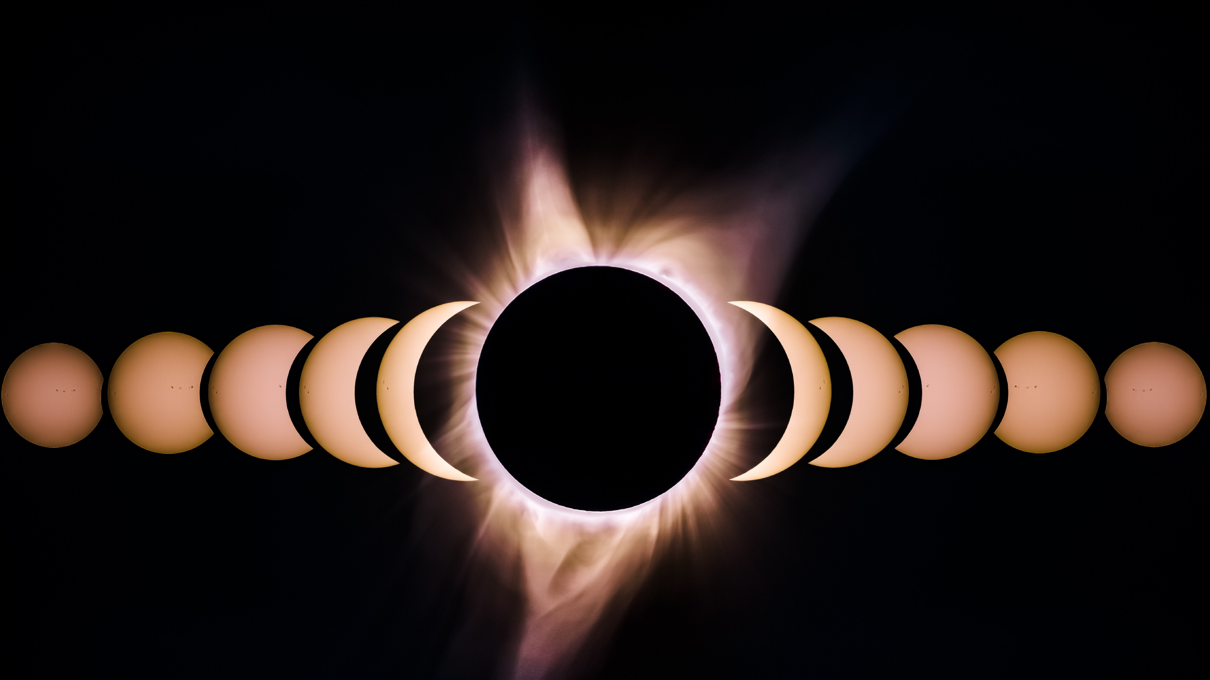
If $f \circ g = 0$, then $\operatorname{im}(g) \subseteq \ker(f)$, and we can form the quotient R-module $\ker(f)/\operatorname{im}(g)$.

Definition

A sequence of such modules A_i and homomorphisms $A_i \xleftarrow{\partial_{i+1}} A_{i+1}$ is called a ***complex of** R**-modules***.

Question

When does $\mathrm{im}(\partial_{i+1}) = \ker(\partial_i)$?



HOW?

Let R be a commutative ring with 1 and let M be an R-module.

Definition

The complex $0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} \cdots$ of free[1] R-modules is called a ***free resolution of*** M provided:

- $M = F_0 / \operatorname{im}(\partial_1)$ and
- $\operatorname{im}(\partial_{i+1}) = \ker(\partial_i)$ for all $i \geq 1$.

In this situation, we call $\ker(\partial_i)$ the ***i-th syzygy of*** M and write $\operatorname{Syz}_i(M)$ when the free resolution is minimal.[2]

---

[1] no relations; a direct sum of copies of R

[2] Minimal: $\partial F_i \subseteq \mathfrak{m} F_{i-1}$ in a local ring (we need this to use NAK to make sure ranks are well-defined)

OUR FIRST FREE RESOLUTION

Let $R = \mathbb{Q}[x]$ and $M = \mathbb{Q}[x]/(x^2)$.

A free resolution of M over R is given by

$$\begin{array}{ccccc} 0 \longleftarrow & F_0 = \mathbb{Q}[x] & \xleftarrow{\partial_1} & F_1 = \mathbb{Q}[x] & \xleftarrow{\partial_2} 0 \\ & x^2 f & \longleftarrow\!\!\shortmid & f \quad 0 & \longleftarrow\!\!\shortmid 0 \end{array}$$

Let's check.

- $M = F_0 / \operatorname{im} \partial_1$?
  - $F_0 = R$ and $\operatorname{im}(\partial_1) = \{x^2 f \mid f \in R\} = (x^2)$
- $\operatorname{im}(\partial_2) = \ker(\partial_1)$?
  - $\operatorname{im}(\partial_2) = 0$ and $\ker(\partial_1) = \{f \in R \mid x^2 f = 0\} = 0$

WHO? WHEN? WHY?

WHY CALCULATE SYZYGIES ANYWAY?

- David Hilbert: Studying invariants at the turn of the 20th century.
- Counting (a way to calculate the Hilbert function of a graded ring)
- Understanding relations among (relations among (relations among (...))) a system of polynomials
- Calculating ranks of Tor or Ext modules.

Definition

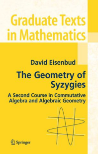
The minimal number of generators of $(\mathrm{Syz}_i(M))_j$ is called the i, j***-th graded Betti number of*** M, denoted $\beta_{i,j}(M)$; we arrange them in a ***Betti table*** denoted by $\beta(M)$. In the table, i indexes columns and j indexes rows.[3]

- Hilbert's Syzygy Theorem: $R = k[x_1, \ldots, x_d] \implies \beta_{d+i,j}(M) = 0$ for all $i \geq 1$
- (Corollary to) Eisenbud's Matrix Factorization Theorem: $R = k[x_1, \ldots, x_d]/(f) \implies \mathrm{Syz}_{d+k}(M) = \mathrm{Syz}_{d+k+2}(M)$ for all $k \geq 1$, and $\beta_{d+1}(M) = \beta_{d+k}(M)$ for all $k \geq 1$
- Boij-Söderberg Theorem: $R = k[x_1, \ldots, x_d] \implies \beta(M) = \sum r_i \beta(\pi_i)$ where $r_i \in \mathbb{Q}_{\geq 0}$, π_i pure
- [G (thesis)] R short Gorenstein ring $\implies \beta(M)$ determined by finite number of columns; upper bound can be determined from the first column

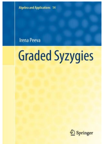
[3] ish...the entry in the i-th column, j-th row is $\beta_{i,j+i}(M)$.

WHERE?

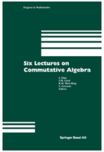
REFERENCES FOR THE INTERESTED ALGEBRAIST



The Geometry of Syzygies, David Eisenbud



Graded Free Resolutions, Irena Peeva



Six Lectures in Commutative Algebra, Chapter 1: Infinite Free Resolutions, Luchezar L. Avramov